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1994 J. Phys. A: Math. Gen. 27 8171

(http://iopscience.iop.org/0305-4470/27/24/024)

J. Phys. A: Math. Gen. 27 (1994) 8171–8183. Printed in the UK

Analytical proof of the random-phase approximation for a model of modulated diffusion

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Received 18 March 1994

Abstract. The random-phase approximation is rigorously proved for a particular model of modulated diffusion consisting in a 1D integrable Hamiltonian map whose phase is modulated by an external Markov noise. Continuity of the diffusion coefficient in the limit of vanishing noise amplitude is also investigated.

1. Introduction

The transport in chaotic regions of Hamiltonian models is a relevant phenomenon in the description of many physical systems [1–7], such as a magnetically confined hot plasma, a particle beam in a magnetic lattice or ring, or a spinning planet or galaxy. Modulated diffusion, in particular, arising from integrable Hamiltonian models whose natural frequencies are modulated by an external time-dependent deterministic or purely random noise, is supposed to be one of the most important contributions to transport in devices for the magnetic confinement of plasmas or particles, and crucial to determine the typical lifetime of these systems [8]. In spite of its practical importance, a mathematical theory of transport is far from being available due to the very complex appearance of phase space, which is crossed, in the typical situation of an integrable Hamiltonian model subjected to a small perturbation, by regular structures such as Kolmogorov–Arnol'd–Moser (KAM) tori, chains of islands and Mather's sets alternating with chaotic domains at infinitely small scales. Invariant manifolds and Mather's sets represent full or partial barriers to transport and may prevent large-scale diffusion or lead to anomalous diffusion rates. In appropriate situations and for systems of dimension larger than two, slow Arnol'd diffusion can occur, even though no rigorous results are yet available for a generic map [8–10]. The problem simplifies somehow in the limit of large perturbation amplitudes, as the regular regions typically come to a very small fraction of the phase-space measure and diffusion can freely develop in an almost ubiquitous chaotic sea. The fast decay of correlations, which is reasonable to expect on the angle variables, allows us to introduce the random-phase approximation (RPA) in studying diffusion on the actions, together with the quasilinear estimate for the corresponding diffusion coefficients [2]. Although physically plausible, the RPA has not yet been justified on an analytical basis even for quite simple 2D symplectic mappings, except for the hyperbolic, continuous algebraic automorphisms of the 2-torus [7].

In more complex models, like a class of almost hyperbolic piecewise linear mappings of the 2-torus lifted to the cylinder [11–14] whose behaviour is very close to billiards [15–18], the existence of a positive, finite diffusion coefficient and stronger statistical properties (central limit theorem, Donsker's invariance principle [19, 20] have been established, but a clear relationship between the diffusion coefficient and the parameters of the map is still missing [12]. A simpler model has recently been proposed in [21, 22] which gives a description of the local diffusion properties of a symplectic map when an external modulation is present. The model is obtained by perturbing an integrable isochronous Hamiltonian map by an external noise. The isochronous hypothesis and the introduction of the external noise, to simulate the effect that in more complicated systems would derive from coupling with hyperbolic degrees of freedom, are the crucial properties which allow one to obtain analytical proofs of existence and positivity for the diffusion coefficient [21–24]. Nevertheless, as numerical investigations suggest, the model is able to reproduce many of the qualitative features noticed in more realistic systems [25], for which analytical results are not available: the RPA in the limit of very large values of the noise amplitude (or perturbation parameter) ε, the oscillating behaviour of the diffusion coefficient D as a function of the same ε, the occurrence of ε domains where D exceeds the corresponding quasilinear estimate D_{ql} (the so-called superlinear regime), and, finally, stronger statistical properties than the mere existence of D (central limit theorem and invariance principle). In this work the analytical proof of the RPA for the above model is discussed, i.e. the convergence of the diffusion coefficient to its quasilinear estimate in the limit $\varepsilon \to \infty$. One of the most interesting aspects of the problem is given by the non-trivial dependence of the diffusion coefficient on the amplitude ε and also by the fact that the latter is a continuous parameter, a quite different situation with respect to the continuous sawtooth map, where the perturbation parameter can only take integer values and the quasilinear estimate holds for all ε [7]. An analytical investigation of the diffusion coefficient trend in the neighborhood of $\varepsilon = 0$ is also presented.

2. A review of the model

We consider a Hamiltonian integrable isochronous map with one degree of freedom in canonical variables and introduce a perturbation on its natural frequency by means of an external deterministic or random noise. A possible choice is given by the Markov modulation $x' = 2x \bmod [0, 1[- \frac{1}{2}$, which leads us to consider the map

$$\begin{cases} \alpha' = 2\alpha \bmod [0,1[- \frac{1}{2} \\ \theta' = \theta + \omega + \varepsilon\alpha \qquad \bmod [0, 2\pi[\\ j' = j + V(\theta) \end{cases} \tag{2.1}$$

where $\theta \in [0, 2\pi[$ and $j \in \mathbb{R}$ are the canonical action-angle variables, $\alpha \in [\frac{1}{2}, \frac{1}{2}[$, endowed with the normalized Haar measure μ_H on the 1-torus $T^1 = [-\frac{1}{2}, \frac{1}{2}[$, $\omega \in \mathbb{R}$ is a constant unperturbed frequency and $\varepsilon \in \mathbb{R}$ a perturbation parameter. Finally, $V(\theta)$ is an arbitrary periodic analytic function of the angle variable θ, with zero mean. Under the technical requirement that $\omega/2\pi$ is a Diophantine number, it is then possible [23] to prove the existence and finiteness of the diffusion coefficient D defined as

$$D = \lim_{n\to+\infty} \frac{E\big((J_{n+1} - J_0)^2\big)}{2(n+1)} \tag{2.2}$$

in which we denote by E the average with respect to the product measure $d\mu(\theta, \alpha) = d\theta \times d\mu_H$ on the space $(\theta, \alpha) \in [0, 2\pi[\times T^1$, and J_n is the value taken at time $n \in \mathbb{N}$ by

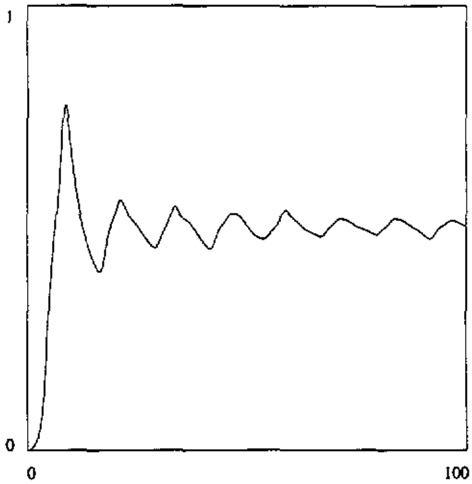

Figure 1. The expression $M(\varepsilon; k, \omega) + \frac{1}{2}$ as a function of the noise amplitude $\varepsilon > 0$, for $k = 1$ and $\omega/2\pi = (\sqrt{5} - 1)/2$, the golden number. For $\varepsilon \gg 0$, the term is very close to $+\frac{1}{2}$, in accordance with the quasilinear estimate.

the invariant associated to the unperturbed mapping [23]. The diffusion coefficient can be written in the form

$$D(\varepsilon) = \frac{1}{2} \sum_{k=-\infty}^{+\infty} |V_k|^2 + \sum_{k=-\infty}^{+\infty} |V_k|^2 M(k, \varepsilon, \omega) \tag{2.3}$$

where $V_k, k \in \mathbb{Z},\ V_0 = 0$, are the Fourier coefficients of the periodic function $V(\theta)$ in the unperturbed map, and the function $M(k, \varepsilon, \omega)$ is defined by

$$M(k, \varepsilon, \omega) = \sum_{d=1}^{\infty} \cos k\omega d \, \frac{\sin \dfrac{k\varepsilon}{2}\left(1 - \dfrac{1}{2^d}\right)}{\dfrac{k\varepsilon}{2}\left(1 - \dfrac{1}{2^d}\right)} \prod_{j=1}^{d} \cos \frac{k\varepsilon}{2}\left(1 - \frac{1}{2^j}\right). \tag{2.4}$$

The expression $D_{\mathrm{ql}} := \frac{1}{2} \sum_{k=-\infty}^{+\infty} |V_k|^2$ represents the quasilinear estimate of the diffusion coefficient [23]. It is possible to show that the above statement reflects good ergodic properties of the reduced skew map [26, 27] obtained from (2.1) by considering the only variables θ and α, whose evolution is independent on the action j [28]. The choice of defining the diffusion coefficient on the invariant of the unperturbed map

$$J(\theta, j) := j - \sum_{k=-\infty}^{+\infty} \frac{V_k}{\mathrm{e}^{ik\omega} - 1} \mathrm{e}^{ik\theta} \tag{2.5}$$

instead of the action j may seem quite cumbersome, but it is suggested by the opportuneness of a sharp distinction between true diffusive behaviour due to frequency modulation and merely ballistic motion concerning integrable dynamics. It is not surprising at all, however, that an analogous result holds for large ε when the diffusion coefficient is directly computed with respect to the action variable.

Notice that from equations (2.3) and (2.4) the existence of a quasilinear limit is far from being trivial; neither is the convergence to zero for $\varepsilon \to 0$, as the rough substitution $\varepsilon = 0$ in (2.3) leads to a meaningless formula. Nevertheless a numerical plot shows both features (figures 1 and 2), and the next sections are devoted to their proof.

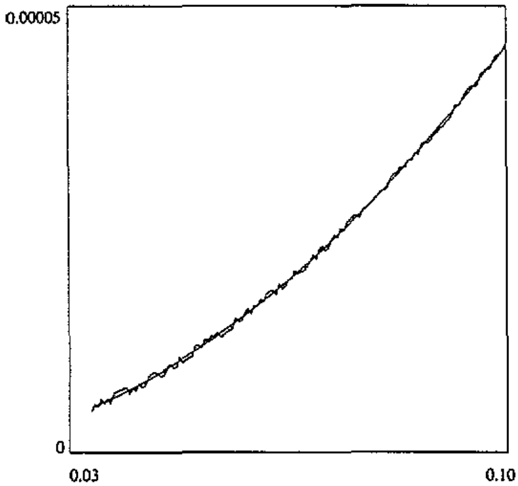

Figure 2. The same as in figure 1, but for smaller values of $\varepsilon > 0$. Truncation errors in the computation of $M(\varepsilon; k, \omega)$ are analytically estimated to be less than 10^{-6}, in agreement with the superposed parabolic fit. The result confirms the quadratic dependence on ε suggested by figure 1.

3. Proof of the random-phase approximation

Our goal is to prove that $\forall k \in \mathbb{Z} \setminus \{0\}$ and $\forall \omega \in \mathbb{R} \setminus \mathbb{Q}$ we have $\lim_{\varepsilon\to\infty} M(k, \varepsilon, \omega) = 0$. Throughout the paper we will assume for simplicity's sake, but with no loss of generality as $M(k, \varepsilon, \omega)$ is an even function of ε, that $\varepsilon > 0$.

First of all we write the expression $M(k, \varepsilon, \omega)$ as a sum of three terms, for which the computation of the limit $\varepsilon \to +\infty$ is relatively simple. $\forall \varepsilon > 0$ and $\forall k, d \in \mathbb{N}$ we introduce the identity

$$\frac{\sin \frac{k\varepsilon}{2}\left(1-\frac{1}{2^d}\right)}{\frac{k\varepsilon}{2}\left(1-\frac{1}{2^d}\right)} = \frac{\sin \frac{k\varepsilon}{2}\left(1-\frac{1}{2^d}\right) - \sin\frac{k\varepsilon}{2}}{\frac{k\varepsilon}{2}\left(1-\frac{1}{2^d}\right)} + \frac{\sin\frac{k\varepsilon}{2}}{\frac{k\varepsilon}{2}} \frac{\frac{1}{2^d}}{1-\frac{1}{2^d}} + \frac{\sin\frac{k\varepsilon}{2}}{\frac{k\varepsilon}{2}} \tag{3.1}$$

and consider the series $E_i(k, \varepsilon, \omega) = \sum_{d=1}^{\infty} C_i(k, \varepsilon, \omega; d)$, $i = 1, 2, 3$, on having defined the coefficients

$$\begin{aligned}
C_1(k,\varepsilon,\omega;d) &:= \cos k\omega d \, \frac{\sin \frac{k\varepsilon}{2}\left(1-\frac{1}{2^d}\right) - \sin\frac{k\varepsilon}{2}}{\frac{k\varepsilon}{2}\left(1-\frac{1}{2^d}\right)} \prod_{j=1}^{d} \cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right) \\
C_2(k,\varepsilon,\omega;d) &:= \cos k\omega d \, \frac{\sin\frac{k\varepsilon}{2}}{\frac{k\varepsilon}{2}} \frac{\frac{1}{2^d}}{1-\frac{1}{2^d}} \prod_{j=1}^{d} \cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right) \\
C_3(k,\varepsilon,\omega;d) &:= \cos k\omega d \, \frac{\sin\frac{k\varepsilon}{2}}{\frac{k\varepsilon}{2}} \prod_{j=1}^{d} \cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right).
\end{aligned} \tag{3.2}$$

We clearly have $M(k, \varepsilon, \omega) = E_1(k, \varepsilon, \omega) + E_2(k, \varepsilon, \omega) + E_3(k, \varepsilon, \omega)$.

Lemma 1. For every $k \in \mathbb{Z} \setminus \{0\}$ and $\forall \omega/2\pi \in \mathbb{R} \setminus \mathbb{Q}$ there holds

$$\lim_{\varepsilon \to +\infty} E_1(k, \varepsilon, \omega) = 0 \qquad \text{and} \qquad \lim_{\varepsilon \to +\infty} E_2(k, \varepsilon, \omega) = 0. \tag{3.3}$$

Proof. We simply have to remark that:
(a) $\forall d, k \in \mathbb{N}$ there exists $\lim_{\varepsilon \to +\infty} C_1(k, \varepsilon, \omega; d) = 0$, as $|C_1(k, \varepsilon, \omega; d)| \leqslant 8/\varepsilon$;
(b) the following upper bound holds:

$$|C_1(k, \varepsilon, \omega; d)| \leqslant |\cos k\omega d| \frac{\left|\sin \dfrac{k\varepsilon}{2}\left(1 - \dfrac{1}{2^d}\right) - \sin \dfrac{k\varepsilon}{2}\right|}{\dfrac{k\varepsilon}{2}\left(1 - \dfrac{1}{2^d}\right)} \leqslant \frac{1}{2^{d-1}} \tag{3.4}$$

and is uniform in $\varepsilon \in \mathbb{R}_+$ and integrable in $d \in \mathbb{N}$, $\sum_{d=1}^{\infty} 1/2^{d-1} = 2$. By applying Lebesgue's dominated convergence theorem we deduce therefore that $\lim_{\varepsilon \to +\infty} E_1(k, \varepsilon, \omega) = 0$. The second part of the lemma follows in an analogous way. □

We now want to find out an upper bound for $|E_3(k, \varepsilon, \omega)|$ which is infinitesimal as $\varepsilon \to +\infty$. To this end we prefix the two following remarks.

Remark 1. By posing, $\forall x \in \mathbb{R}$, $\text{dist}(x, \pi\mathbb{Z}) := \inf_{p \in \mathbb{Z}} |x - p\pi|$ and taking $j \in \mathbb{N}$ such that

$$j > \overline{j}(k\varepsilon) := \frac{1}{\log 2} \log\left[\frac{\frac{k\varepsilon}{2}}{\text{dist}\left(\frac{k\varepsilon}{2}, \pi\mathbb{Z}\right)}\right] \tag{3.5}$$

there holds $k\varepsilon/2^{j+1} < \text{dist}\left(\frac{k\varepsilon}{2}, \pi\mathbb{Z}\right)$ and finally $\text{sgn} \cos \frac{k\varepsilon}{2} = \text{sgn} \cos \frac{k\varepsilon}{2}$.

Remark 2. For every $x \in \mathbb{R}_+$ we have

$$\left|\frac{\sin x}{x} \log\left[\frac{x}{\text{dist}(x, \pi\mathbb{Z})}\right]\right| \leqslant \sup_{\alpha \in \mathbb{R}} \left|\frac{\sin \alpha}{\alpha}\right| \left|\frac{\log\left(\dfrac{x}{\text{dist}(x, \pi\mathbb{Z})}\right)}{\dfrac{x}{\text{dist}(x, \pi\mathbb{Z})}}\right| = \left|\frac{\log\left(\dfrac{x}{\text{dist}(x, \pi\mathbb{Z})}\right)}{\dfrac{x}{\text{dist}(x, \pi\mathbb{Z})}}\right|. \tag{3.6}$$

As $|\log \beta/\beta|$ is a monotonic decreasing function in the interval $\beta \in [e, +\infty[$, we deduce that $\forall \varepsilon \geqslant \pi e$ there holds

$$\left|\frac{\sin \frac{k\varepsilon}{2}}{\frac{k\varepsilon}{2}} \log\left[\frac{\frac{k\varepsilon}{2}}{\text{dist}\left(\frac{k\varepsilon}{2}, \pi\mathbb{Z}\right)}\right]\right| \leqslant \sup_{\beta \geqslant \varepsilon/\pi} \left|\frac{\log \beta}{\beta}\right| = \left|\frac{\log \varepsilon/\pi}{\varepsilon/\pi}\right| \tag{3.7}$$

because $\frac{k\varepsilon}{2}/\text{dist}\left(\frac{k\varepsilon}{2}, \pi\mathbb{Z}\right) \geqslant e \; \forall k \in \mathbb{N}$.

We can now state the following:

Lemma 2. Let $\gamma_\omega > 0$ and $\mu_\omega \geqslant 2$ be the characteristic parameters of the Diophantine frequency $\omega/2\pi$, so that

$$\left|\frac{\omega}{2\pi} - \frac{p}{q}\right|^{-1} \leqslant \gamma_\omega q^{\mu_\omega} \qquad \forall p \in \mathbb{Z},\ q \in \mathbb{N}. \tag{3.8}$$

Then, $\forall \varepsilon/2\pi \in \mathbb{R}_+ \setminus \mathbb{Q}$, $\varepsilon/2\pi > e/2$, and $\forall k \in \mathbb{N}$, the following upper bound holds:

$$|E_3(k,\varepsilon,\omega)| \leqslant \frac{1}{\log 2}\left|\frac{\log \varepsilon/\pi}{\varepsilon/\pi}\right| + 2^{\mu_\omega - 1}\gamma_\omega k^{\mu_\omega - 2}\frac{1}{\varepsilon}. \tag{3.9}$$

Proof. We preliminarily write the inequality

$$|E_3(k,\varepsilon,\omega)| \leqslant \frac{1}{\log 2}\left|\frac{\log \varepsilon/\pi}{\varepsilon/\pi}\right| + \left|\sum_{d=\overline{j}(k\varepsilon)+1}^{\infty} C_3(k,\varepsilon,\omega;d)\right|. \tag{3.10}$$

By using remark 1, the residual series admits the further bound

$$\left|\sum_{d=\overline{j}(k\varepsilon)+1}^{\infty} \cos k\omega d \left[\operatorname{sgn}\cos\frac{k\varepsilon}{2}\right]^{d-\overline{j}(k\varepsilon)} \prod_{j=\overline{j}(k\varepsilon)+1}^{d}\left|\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\right|\right|\left|\frac{\sin\frac{k\varepsilon}{2}}{\frac{k\varepsilon}{2}}\right|. \tag{3.11}$$

From Abel's inequality we get, $\forall n \in \mathbb{N}$, $n > \overline{j}(k\varepsilon) + 2$:

$$\begin{aligned}&\left|\sum_{d=\overline{j}(k\varepsilon)+1}^{n-1} \cos k\omega d \left[\operatorname{sgn}\cos\frac{k\varepsilon}{2}\right]^{d-\overline{j}(k\varepsilon)} \prod_{j=\overline{j}(k\varepsilon)+1}^{d}\left|\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\right|\right| \\ &\qquad \leqslant \operatorname*{Sup}_{s\in\{\overline{j}(k\varepsilon)+1,\dots,n-1\}}\left|\sum_{d=\overline{j}(k\varepsilon)+1}^{s} \cos k\omega d\left[\operatorname{sgn}\cos\frac{k\varepsilon}{2}\right]^d\right| \leqslant 2^{\mu_\omega-1}\gamma_\omega k^{\mu_\omega-1}\end{aligned} \tag{3.12}$$

independent on n, and inserting in (3.10) we deduce the lemma. □

We can state a similar result concerning the case $\varepsilon/2\pi \in \mathbb{Q}_+$.

Lemma 3. With the same notations used in lemma 2, $\forall\varepsilon/2\pi \in \mathbb{Q}_+$ there holds

$$|E_3(k,\varepsilon,\omega)| \leqslant \frac{1}{\log 2}\frac{\log(\varepsilon/\pi)}{\varepsilon/\pi} + 2^{\mu_\omega}\gamma_\omega k^{\mu_\omega-2}\frac{1}{\varepsilon}. \tag{3.13}$$

Proof. Let $\varepsilon = 2\pi p/q$, with $p, q \in \mathbb{N}$ and relatively prime. Expression $E_3(k,\varepsilon,\omega)$ then becomes

$$\sum_{d=1}^{\infty}\cos k\omega d\frac{\sin\pi\frac{pk}{q}}{\pi\frac{pk}{q}}\prod_{j=1}^{d}\cos\pi\frac{pk}{q}\left(1\frac{1}{2^j}\right). \tag{3.14}$$

If $pk/q \neq h + \frac{1}{2}$, $h \in \mathbb{N}\cup\{0\}$, we can repeat the proof of the previous lemma 2, as $\cos(\pi pk/q) \neq 0$. On the contrary, let $pk/q = h + \frac{1}{2}$, for some $h \in \mathbb{N}\cup\{0\}$. As a consequence $\sin(\pi pk/q) = (-1)^h$, whereas

$$\cos\pi\frac{pk}{q}\left(1-\frac{1}{2^j}\right) = (-1)^h\sin\left[\pi\left(h+\tfrac{1}{2}\right)\frac{1}{2^j}\right]. \tag{3.15}$$

Therefore

$$|E_3(k,\varepsilon,\omega)| = \frac{1}{\pi\left(\frac{1}{2}+h\right)}\left|\sum_{d=1}^{\infty}\cos k\omega d\,(-1)^{hd}\prod_{j=1}^{d}\sin\left[\pi\left(h+\tfrac{1}{2}\right)\frac{1}{2^j}\right]\right|. \tag{3.16}$$

By taking $j \in \mathbb{N}$ such that $j > \hat{j}(h) := \log(1+2h)/\log 2$ we have $\sin\left[\pi\left(h+\frac{1}{2}\right)\frac{1}{2^j}\right] > 0$ and we can bound the previous expression by

$$\frac{\frac{2}{\log 2}\log(1+2h)}{\pi(1+2h)} + \frac{1}{\pi\left(\frac{1}{2}+h\right)}\left|\sum_{\hat{j}(h)+1}^{\infty}\cos k\omega d(-1)^{hd}\prod_{j=\hat{j}(h)+1}^{d}\sin\left[\pi\left(h+\tfrac{1}{2}\right)\frac{1}{2^j}\right]\right|. \tag{3.17}$$

Abel's inequality allows us to write, $\forall n \in \mathbb{N}$, $n \geqslant \hat{j}(h)+1$, the upper bound

$$\left|\sum_{\hat{j}(h)+1}^{n-1}\cos k\omega d(-1)^{hd}\prod_{j=\hat{j}(h)+1}^{d}\sin\left[\pi\left(h+\tfrac{1}{2}\right)\frac{1}{2^j}\right]\right| \leqslant 2^{\mu_\omega-1}\gamma_\omega k^{\mu_\omega-1} \tag{3.18}$$

because of the inequality

$$\left|\sum_{d=\hat{j}(h)+1}^{s}\cos k\omega d(-1)^{hd}\right| \leqslant 2^{\mu_\omega-1}\gamma_\omega k^{\mu_\omega-1}.$$

As a conclusion, keeping in mind that $\varepsilon k/\pi = 1+2h \geqslant 1$, equation (3.17) becomes

$$|E_3(k,\varepsilon,\omega)| \leqslant \frac{2}{\pi\log 2}\frac{\log(\varepsilon k/\pi)}{\varepsilon k/\pi} + 2^{\mu_\omega}\gamma_\omega k^{\mu_\omega-2}\frac{1}{\varepsilon}. \tag{3.19}$$

Moreover, $\forall k \in \mathbb{N}$ it is $\varepsilon k/\pi \geqslant \varepsilon/\pi$ and if we take $\varepsilon \geqslant \pi e$ we get

$$\frac{\log(\varepsilon k/\pi)}{\varepsilon k/\pi} \leqslant \frac{\log(\varepsilon/\pi)}{\varepsilon/\pi}$$

and whence

$$\frac{2}{\pi\log 2}\frac{\log(\varepsilon k/\pi)}{\varepsilon k/\pi} < \frac{1}{\log 2}\frac{\log(\varepsilon/\pi)}{\varepsilon/\pi}$$

which completes the proof. □

By collecting the results of lemmas 2 and 3 we conclude that the bound (3.13) can be applied both in the case of $\varepsilon/2\pi \in \mathbb{R}_+ \setminus \mathbb{Q}$ and in the case of $\varepsilon/2\pi \in \mathbb{Q}_+$, $\forall k \in \mathbb{N}$ and $\forall\varepsilon > e\pi$. It follows that $\lim_{\varepsilon\to+\infty} E_3(k,\varepsilon,\omega) = 0$. This motivates the asymptotic behaviour of the plot in figure 1, which for every $k \in \mathbb{N}$ tends to the limit $\lim_{\varepsilon\to+\infty} E_1(k,\varepsilon,\omega)+E_2(k,\varepsilon,\omega)+E_3(k,\varepsilon,\omega)+\frac{1}{2} = \frac{1}{2}$. Now we have all the elements to prove the RPA in the general case. The correction to the quasilinear estimate of the diffusion coefficient can be written in the form

$$\sum_{k\neq 0}|V_k|^2 E_1(k,\varepsilon,\omega) + \sum_{k\neq 0}|V_k|^2 E_2(k,\varepsilon,\omega) + \sum_{k\neq 0}|V_k|^2 E_3(k,\varepsilon,\omega). \tag{3.20}$$

Let us then state the following:

Theorem 1 (Random-phase approximation). For every $\omega/2\pi$ Diophantine and $\forall V(\theta)$, 2π-periodic analytic function of the angle θ, with Fourier coefficients V_k, $k \in \mathbb{Z}$ and zero mean, there holds

$$\lim_{\varepsilon\to+\infty} \sum_{k\neq 0} |V_k|^2 E_1(k,\varepsilon,\omega) + \sum_{k\neq 0} |V_k|^2 E_2(k,\varepsilon,\omega) + \sum_{k\neq 0} |V_k|^2 E_3(k,\varepsilon,\omega) = 0 \tag{3.21}$$

so that the limit $\varepsilon \to \infty$ of the diffusion coefficient coincides with the quasilinear estimate D_{ql}.

Proof. By the symmetry on k we can write that

$$\sum_{k\neq 0} |V_k|^2 E_1(k,\varepsilon,\omega) = 2\sum_{k=1}^{\infty}\sum_{d=1}^{\infty} |V_k|^2 C_1(k,\varepsilon,\omega;d). \tag{3.22}$$

Moreover, as we have already shown, $\lim_{\varepsilon\to+\infty} |V_k|^2 C_1(k,\varepsilon,\omega;d) = 0$, $\forall k,d \in \mathbb{N}$, whereas

$$\left||V_k|^2 C_1(k,\varepsilon,\omega;d)\right| \leqslant |V_k|^2 \frac{1}{2^{d-1}} \tag{3.23}$$

which is uniform in ε and integrable with respect to $(k,d) \in \mathbb{N}\times\mathbb{N}$, because of the exponential decay of the Fourier coefficients V_k. By the dominated convergence theorem one can deduce that (3.22) converges to zero as $\varepsilon \to +\infty$. In an analogous way we can prove that $\lim_{\varepsilon\to+\infty}\sum_{k\neq 0} |V_k|^2 E_2(k,\varepsilon,\omega) = 0$, as inequality (3.23) holds as well. Finally, let us consider

$$\left|\sum_{k\neq 0} |V_k|^2 E_3(k,\varepsilon,\omega)\right| \leqslant 2\sum_{k=1}^{\infty} |V_k|^2 |E_3(k,\varepsilon,\omega)| \tag{3.24}$$

which can be bounded $\forall\varepsilon > \pi e$, according to (3.13), by

$$\frac{2}{\log 2}\sum_{k=1}^{\infty} |V_k|^2 \frac{\log(\varepsilon/\pi)}{\varepsilon/\pi} + 2^{\mu_\omega+1}\gamma_\omega \sum_{k=1}^{\infty} |V_k|^2 k^{\mu_\omega-2}\frac{1}{\varepsilon} \tag{3.25}$$

and where, again, the series are convergent because of the exponential decay of the Fourier coefficients. Therefore $\lim_{\varepsilon\to+\infty}\sum_{k\neq 0}|V_k|^2 E_3(k,\varepsilon,\omega) = 0$ and the proof is complete. □

4. Continuity at $\varepsilon = 0$ of the diffusion coefficient

For simplicity's sake we confine ourselves to the case in which $V(\theta)$ is a trigonometric polynomial, so that the Fourier spectrum V_k is finite. What we have to show is therefore that $\lim_{\varepsilon\to 0} M(k,\varepsilon,\omega) = -\frac{1}{2}$ $\forall k \in \mathbb{Z}\setminus\{0\}$ and $\forall\omega \in \mathbb{R}\setminus\mathbb{Q}$. Due to the symmetry of the function $M(k,\varepsilon,\omega)$ with respect to ε, we may compute the only limit from the right.

Lemma 4. $\forall k \in \mathbb{Z} \setminus \{0\}$ and $\forall \omega/2\pi \in \mathbb{R} \setminus \mathbb{Q}$ there holds $\lim_{\varepsilon\to 0} M(k, \varepsilon, \omega) = \lim_{\varepsilon\to 0} N(k, \varepsilon, \omega)$ provided that the limit on the RHS exists, on having defined the expression

$$N(k,\varepsilon,\omega) := \sum_{d=1}^{\infty} \left[\frac{\sin k\omega\left(d+\frac{1}{2}\right)}{2\sin\frac{k\omega}{2}} - \frac{1}{2} \right] \prod_{j=1}^{d} \cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right) \left[1-\cos\frac{k\varepsilon}{2}\right]. \tag{4.1}$$

Proof. First of all we remove the correction $1/2^d$ in the argument of the function $\sin x/x$. A straightforward application of the dominated convergence theorem shows that the series

$$\sum_{d=1}^{\infty} \cos k\omega d \left[\frac{\sin\frac{k\varepsilon}{2}\left(1-\frac{1}{2^d}\right)}{\frac{k\varepsilon}{2}\left(1-\frac{1}{2^d}\right)} - \frac{\sin\frac{k\varepsilon}{2}}{\frac{k\varepsilon}{2}} \right] \prod_{j=1}^{d} \cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right) \tag{4.2}$$

tends to zero as $\varepsilon \to 0+$. Therefore, we simply have to discuss the residual limit

$$\begin{aligned}
\lim_{\varepsilon\to 0+} \frac{\sin\frac{k\varepsilon}{2}}{\frac{k\varepsilon}{2}} \sum_{d=1}^{\infty} \cos k\omega d \prod_{j=1}^{d} \cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right) \\
= \lim_{\varepsilon\to 0+} \sum_{d=1}^{\infty} \cos k\omega d \prod_{j=1}^{d} \cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right).
\end{aligned} \tag{4.3}$$

To this end, let us rewrite the series

$$\hat{M}(k,\varepsilon,\omega) := \sum_{d=1}^{\infty} \cos k\omega d \prod_{j=1}^{d} \cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)$$

in a more suitable form for the computation of the limit $\varepsilon \to 0+$, by integrating by parts (Brunacci–Abel' identity). Let us preliminarily pose

$$F_d(k\omega) := \begin{cases} \displaystyle\sum_{s=1}^{d} \cos k\omega s = \Re\mathrm{e}\left[\mathrm{e}^{\mathrm{i}k\omega} \frac{1-\mathrm{e}^{\mathrm{i}k\omega d}}{1-\mathrm{e}^{\mathrm{i}k\omega}} \right] & \forall d \in \mathbb{N} \\ 0 & \text{if } d = 0 \end{cases} \tag{4.4}$$

so that $\cos k\omega d = F_d(k\omega) - F_{d-1}(k\omega)\ \forall d \in \mathbb{N}$. Thus

$$\begin{aligned}
\hat{M}(k,\varepsilon,\omega) &= \sum_{d=1}^{\infty} [F_d(k\omega) - F_{d-1}(k\omega)] \\
&= \lim_{N\to+\infty} F_N(k\omega) \prod_{j=1}^{N} \cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right) - F_0(k\omega)\frac{k\varepsilon}{4} \\
&\quad + \sum_{d=1}^{N-1} F_d(k\omega) \prod_{j=1}^{d} \left[1-\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^{d+1}}\right)\right].
\end{aligned} \tag{4.5}$$

If $0 < k\varepsilon/2 < \pi/2$, there holds $\lim_{N\to+\infty}\prod_{j=1}^{N}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right) = 0$ and, as $|F_N(k\omega)| \leqslant 2/|1-\mathrm{e}^{\mathrm{i}k\omega}|$, we conclude

$$\hat{M}(k,\varepsilon,\omega) = \sum_{d=1}^{\infty} F_d(k\omega)\prod_{j=1}^{d}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\left[1-\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^{d+1}}\right)\right]. \tag{4.6}$$

The term $1/2^{d+1}$ in the previous expression can be dropped, by introducing an error of order $o(\varepsilon)$, as $\varepsilon \to 0+$. Indeed, we now have

$$\left|\sum_{d=1}^{\infty} F_d(k\omega)\prod_{j=1}^{d}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\left[\cos\frac{k\varepsilon}{2}-\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^{d+1}}\right)\right]\right| \leqslant \frac{1}{|\mathrm{e}^{\mathrm{i}\omega}-1|}\left|\frac{k\varepsilon}{2}\sin\frac{k\varepsilon}{2}\right| \tag{4.7}$$

where we used $|\sin x/x| \leqslant 1\ \forall x \in \mathbb{R}$. The final expression is evidently $o(\varepsilon)$, as $\varepsilon \to 0+$. By introducing the Dirichlet kernel

$$\frac{1}{2}+\sum_{j=1}^{d}\cos jk\omega = \frac{\sin k\omega\left(d+\frac{1}{2}\right)}{2\sin\frac{k\omega}{2}} \qquad \forall k\omega \in \mathbb{R}\setminus\{0\}, d \in \mathbb{N} \tag{4.8}$$

we can come to deal with the $\lim_{\varepsilon\to0+}$ of the series

$$N(k,\varepsilon,\omega) := \sum_{d=1}^{\infty}\left[\frac{\sin k\omega\left(d+\frac{1}{2}\right)}{2\sin\frac{k\omega}{2}}-\frac{1}{2}\right]\prod_{j=1}^{d}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\left[1-\cos\frac{k\varepsilon}{2}\right]. \tag{4.9}$$

□

Lemma 5. In the same hypotheses of lemma 4 we have the identity

$$\lim_{\varepsilon\to0+} N(k,\varepsilon,\omega) = -\frac{1}{2}+\lim_{\varepsilon\to0+}\sum_{d=1}^{\infty}\frac{\sin k\omega\left(d+\frac{1}{2}\right)}{2\sin\frac{k\omega}{2}}\prod_{j=1}^{d}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\left[1-\cos\frac{k\varepsilon}{2}\right]. \tag{4.10}$$

provided that the limit on the right-hand side exists.

Proof. We have to show that

$$\lim_{\varepsilon\to0+}\sum_{d=1}^{\infty}\prod_{j=1}^{d}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\left[1-\cos\frac{k\varepsilon}{2}\right] = 1. \tag{4.11}$$

Consider $k\varepsilon/2 < \pi/2$ and take a fixed $n \in \mathbb{N}$. We then have, $\forall j \geqslant n$, $j \in \mathbb{N}$ and $\forall d \geqslant n$

$$\cos^{d-n+1}\frac{k\varepsilon}{2}\left(1-\frac{1}{2^n}\right) \geqslant \prod_{j=n}^{d}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right) > \cos^{d-n+1}\frac{k\varepsilon}{2} > 0. \tag{4.12}$$

Further, by posing

$$\sum_{d=1}^{\infty}\prod_{j=1}^{d}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)=\sum_{d=1}^{n-1}\prod_{j=1}^{d}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)+\sum_{d=n}^{\infty}\prod_{j=1}^{d}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right) \tag{4.13}$$

the series on the right-hand side admits the bounds

$$\begin{aligned}\prod_{j=1}^{n-1}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\sum_{d=n}^{\infty}\cos^{d-n+1}\frac{k\varepsilon}{2}<\sum_{d=n}^{\infty}\prod_{j=1}^{d}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\\ \leqslant\prod_{j=1}^{n-1}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\sum_{d=n}^{\infty}\cos^{d-n+1}\frac{k\varepsilon}{2}\left(1-\frac{1}{2^n}\right).\end{aligned} \tag{4.14}$$

As a consequence, $\forall k\varepsilon/2<\pi/2$ the expression

$$S(k\varepsilon):=\left[1-\cos\frac{k\varepsilon}{2}\right]\sum_{d=1}^{\infty}\prod_{j=1}^{d}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right) \tag{4.15}$$

can be controlled from above by

$$\begin{aligned}&\sum_{j=1}^{n-1}\prod_{j=1}^{d}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\left[1-\cos\frac{k\varepsilon}{2}\right]\\ &\quad+\prod_{j=1}^{n-1}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\left[1-\cos\frac{k\varepsilon}{2}\right]\frac{\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^n}\right)}{1-\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^n}\right)}\end{aligned} \tag{4.16a}$$

and from below by

$$\sum_{j=1}^{n-1}\prod_{j=1}^{d}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\left[1-\cos\frac{k\varepsilon}{2}\right]+\prod_{j=1}^{n-1}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\cos\frac{k\varepsilon}{2} \tag{4.16b}$$

for any fixed $n\in\mathbb{N}$. The $\lim_{\varepsilon\to0+}$ of (4.16*a*) can immediately be written as

$$\lim_{\varepsilon\to0+}\frac{2\sin^2\frac{k\varepsilon}{4}}{2\sin^2\frac{k\varepsilon}{4}\left(1-\frac{1}{2^n}\right)}=\frac{1}{\left(1-\frac{1}{2^n}\right)^2} \tag{4.17}$$

and that of (4.16*b*) as

$$\lim_{\varepsilon\to0+}\prod_{j=1}^{n-1}\cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\cos\frac{k\varepsilon}{2}=1.$$

As a conclusion, $\forall n\in\mathbb{N}$ there holds

$$1\leqslant\liminf_{\varepsilon\to0+}S(k\varepsilon)\leqslant\limsup_{\varepsilon\to0+}S(k\varepsilon)\leqslant\left(1-\frac{1}{2^n}\right)^{-2}$$

and from the arbitrariness of $n\in\mathbb{N}$ we conclude that $\lim_{\varepsilon\to0+}S(k\varepsilon)=1$, which completes the proof of the lemma. □

Lemma 6. In the same hypotheses of lemma 4 we have that

$$\lim_{\varepsilon\to 0+} \sum_{d=1}^{\infty} \frac{\sin k\omega\,(d+\frac{1}{2})}{2\sin\frac{k\omega}{2}} \prod_{j=1}^{d} \cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\left[1-\cos\frac{k\varepsilon}{2}\right] = 0 \tag{4.18}$$

Proof. We have already proved that for $0 < \frac{k\varepsilon}{2} < \frac{\pi}{2}$ there holds $1 > \cos\frac{k\varepsilon}{2}(1-\frac{1}{2^j}) > 0$ $\forall j \in \mathbb{N}$. Therefore

$$\left\{\prod_{j=1}^{d} \cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\right\}_{d\in\mathbb{N}} \tag{4.19}$$

is a positive, decreasing sequence. Thus, from Abel's inequality we deduce, $\forall n \in \mathbb{N}$,

$$\begin{aligned}\left|\sum_{d=1}^{n} \sin k\omega\,(d+\tfrac{1}{2}) \prod_{j=1}^{d} \cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\right| &\leqslant \cos\frac{k\varepsilon}{4} \operatorname*{Sup}_{s=1,2,\ldots,n} \left|\sum_{d=1}^{s} \sin k\omega\,(d+\tfrac{1}{2})\right| \\ &\leqslant \frac{2}{|\mathrm{e}^{\mathrm{i}k\omega}-1|}.\end{aligned} \tag{4.20}$$

Hence

$$\left|\sum_{d=1}^{\infty} \sin k\omega(d+\tfrac{1}{2}) \prod_{j=1}^{d} \cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\right| \leqslant \frac{2}{|\mathrm{e}^{\mathrm{i}k\omega}-1|} \tag{4.21}$$

and as a consequence

$$\left|\sum_{d=1}^{\infty} \frac{\sin k\omega\,(d+\frac{1}{2})}{2\sin\frac{k\omega}{2}} \prod_{j=1}^{d} \cos\frac{k\varepsilon}{2}\left(1-\frac{1}{2^j}\right)\left[1-\cos\frac{k\varepsilon}{2}\right]\right| \leqslant \frac{1}{2\sin^2\frac{k\omega}{2}}\left[1-\cos\frac{k\varepsilon}{2}\right] \tag{4.22}$$

which tends to zero as $\varepsilon \to 0+$. □

A straightforward application of lemmas 4–6 now enables us to state the final theorem:

Theorem 2. For any trigonometric polynomial $V(\theta)$ the diffusion coefficient $D(\varepsilon)$ satisfies the following continuity condition at $\varepsilon = 0$:

$$\lim_{\varepsilon\to 0} D(\varepsilon) = 0. \tag{4.23}$$

Theorem 2 is equivalent to stating that for vanishing modulation amplitude ε the diffusion coefficient tends to zero, which is to be expected by physical considerations—vanishing modulation means constant J and therefore vanishing diffusion coefficient. Numerical simulations also suggest that the dependence on ε is quadratic in the neighbourhood of 0 (see figure 2).

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